SPECIALIZATION OF MOTIVIC HODGE-CHERN CLASSES

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ABSTRACT. In this paper we give a proof of the fact, that the motivic Hodge-Chern class transformation MHC_y and Hirzebruch class transformation MHT_{y*} for mixed Hodge modules and strictly specializable filtered $\mathcal{D}$-modules commute with specialization in the algebraic and in a suitable complex analytic context. Here specialization in the Hodge- and $\mathcal{D}$-module context means the corresponding nearby cycles defined in terms of the V-filtration of Malgrange-Kashiwara. This generalizes a corresponding specialization result of Verdier about MacPherson's Chern class transformation c_*.

1. INTRODUCTION

Let $k \subset \mathbb{C}$ be a subfield of the complex numbers. Using Saito's deep theory of *algebraic mixed Hodge modules* over k ([18]-[24]), we introduced in [4] the *motivic Chern and Hirzebruch class transformations mC_y, MHC_y and T_{y*}, MHT_{y*}* as natural transformations (cummuting with push down for proper morphisms) fitting into a commutative diagram (with $\mathbb{Q}[y]_{loc} := \mathbb{Q}[y, y^{-1}, (1+y)^{-1}]$):

$$
\begin{array}{ccccc}
G_0(X)[y] & \longrightarrow & G_0(X)[y, y^{-1}] & = & G_0(X)[y, y^{-1}] \\
\big\uparrow{\scriptstyle mC_y} & & \big\uparrow{\scriptstyle mC_y} & & \big\uparrow{\scriptstyle MHC_y} \\
K_0(var/X) & \longrightarrow & \mathcal{M}(var/X) & \xrightarrow{\chi_{Hdg}} & K_0(MHM(X/k)) \\
\big\downarrow{\scriptstyle T_{y*}} & & \big\downarrow{\scriptstyle T_{y*}} & & \big\downarrow{\scriptstyle MHT_{y*}} \\
H_*(X) \otimes \mathbb{Q}[y] & \longrightarrow & H_*(X) \otimes \mathbb{Q}[y, y^{-1}] & \longrightarrow & H_*(X) \otimes \mathbb{Q}[y]_{loc} \,.
\end{array}
$$

Here $G_0(X)$ resp. $K_0(MHM(X/k))$ is the Grothendieck group of coherent sheaves resp. algebraic mixed Hodge modules on X, $K_0(var/X)$ is the relative Grothendieck group af algebraic k-varieties over X, with $\mathcal{M}(var/X) := K_0(var/X)[\mathbb{L}^{-1}]$ its localization with respect to the class of the affine line

$$
\mathbb{L} = [\mathbb{A}^1_k \to pt] \in K_0(var/pt)
$$

(compare e.g. [2]). and finally $H_*(X) = CH_*(X)$ is the Chow homology group [7], or for $k = \mathbb{C}$, $H_*(X) = H^{BM}_{2*}(X)$ can also be the Borel-Moore homology of X (in even degrees).

The motivic Chern class transformations mC_y, MHC_y are a K-theoretical refinement of the Hirzebruch class transformations T_{y*}, MHT_{y*}, since they

are related by a (functorial) commutative diagram :

$$\begin{array}{ccccc}
\mathcal{M}(var/X) & \xrightarrow{\chi_{Hdg}} & K_0(MHM(X/k)) & \xrightarrow{MHC_y} & G_0(X)[y,y^{-1}] \\
{\scriptstyle T_{y*}}\downarrow & & {\scriptstyle MHT_{y*}}\downarrow & & \downarrow{\scriptstyle td*} \\
H_*(X)\otimes\mathbb{Q}[y,y^{-1}] & \longrightarrow & H_*(X)\otimes\mathbb{Q}_{loc} & \xleftarrow{(1+y)^{-*}\cdot} & H_*(X)\otimes\mathbb{Q}[y,y^{-1}]\,,
\end{array}$$

with $td_* : G_0(X) \to H_*(X)\otimes\mathbb{Q}$ the *Todd class transformation* of Baum-Fulton-MacPherson [1, 7] and $(1+y)^{-*}\cdot$ the renormalization given in degree i by the multiplication

$$(1+y)^{-i}\cdot : H_i(-)\otimes\mathbb{Q}[y,y^{-1}] \to H_i(-)\otimes\mathbb{Q}[y,y^{-1},(1+y)^{-1}]\,.$$

The characteristic class transformations mC_y, T_{y*} are motivic refinements of the (rationalization of the) *Chern class transformation*

$$c_* : F(X) \to H_*(X)$$

of MacPherson [13, 10], with $F(X)$ the abelian group of algebraically constructible functions, fitting into a (functorial) commutative diagram

$$\begin{array}{ccc}
F(X) & \xleftarrow{\quad can \quad} & \mathcal{M}(var/X) \\
{\scriptstyle c_*\otimes\mathbb{Q}}\downarrow & & \downarrow{\scriptstyle T_{y*}} \\
H_*(X)\otimes\mathbb{Q} & \xleftarrow{\quad y=-1 \quad} & H_*(X)\otimes\mathbb{Q}[y,y^{-1}]\,.
\end{array}$$

For $k = \mathbb{C}$, this was further improved in our recent survey [27][Prop.5.2.1]. MHT_{y*} factorizes as

$$MHT_{y*} : K_0(MHM(X)) \to H_*(X)\otimes\mathbb{Q}[y,y^{-1}] \subset H_*(X)\otimes\mathbb{Q}[y]_{loc}\,,$$

fitting into a (functorial) commutative diagram

$$\begin{array}{ccccc}
(1) & F(X) & \xleftarrow{\chi_{stalk}} K_0(D_c^b(X)) & \xleftarrow{\quad rat \quad} & K_0(MHM(X)) \\
& {\scriptstyle c_*\otimes\mathbb{Q}}\downarrow & {\scriptstyle c_*\otimes\mathbb{Q}}\downarrow & & \downarrow{\scriptstyle MHT_{y*}} \\
& H_*(X)\otimes\mathbb{Q} = \!\!= H_*(X)\otimes\mathbb{Q} & \xleftarrow{\quad y=-1 \quad} & H_*(X)\otimes\mathbb{Q}[y,y^{-1}]\,.
\end{array}$$

Here $D_c^b(X)$ is the derived category of algebraically constructible sheaves (with rational coefficients) on X (viewed as a complex analytic space), with

$$rat : D^bMHM(X) \to D_c^b(X) \quad \text{and} \quad rat : MHM(X) \to Perv(X)$$

associating to a (complex of) mixed Hodge module(s) the underlying (perverse) constructible sheaf complex. Then

$$K_0(D_c^b(X)) = K_0(Perv(X)) \quad \text{and} \quad K_0(D^bMHM(X)) = K_0(MHM(X))$$

(see e.g. [26][lem.3.3.1]), and χ_{stalk} is given by the Euler characteristic of the stalks. Note that

$$can : K_0(var/X) \to F(X) \quad \text{and} \quad \chi_{stalk}\circ rat : K_0(MHM(X)) \to F(X)$$

are surjective. Moreover, important functoriality results known for the MacPherson Chern class transformation c_*, like

(1) functoriality for proper morphisms,
(2) multiplicativity for exterior products, and
(3) a Verdier Riemann-Roch formula for smooth pullbacks,

have been extended to mC_y, T_{y*} in [4] and to MHC_y, MHT_{y*} in [27].

The aim of this paper is to prove for these motivic transformations the counterpart of a famous result of Verdier, that in the complex context the MacPherson Chern class transformation c_* *commutes with specialization* [29]. Let $f : X \to \mathbb{C}$ an algebraic function with $X_0 := \{f = 0\}$. Then Deligne's *nearby and vanishing cycle functors* (compare e.g. [26])

$$\Psi_f, \Phi_f : D_c^b(X) \to D_c^b(X_0)$$

induce similar transformations for constructible functions fitting into a commutative diagram

(2)
$$
\begin{array}{ccc}
K_0(D_c^b(X)) & \xrightarrow{\Psi_f, \Phi_f} & K_0(D_c^b(X_0)) \\
\chi_{stalk} \downarrow & & \chi_{stalk} \downarrow \\
F(X) & \xrightarrow{\Psi_f, \Phi_f} & F(X_0) \,.
\end{array}
$$

See also [25] for an elementary description (without using sheaf theory) of $\Psi_f, \Phi_f : F(X) \to F(X_0)$ in terms of local *Milnor fibrations*. Assume now that X_0 is a hypersurface of codimension one, so that one also has a homological *Gysin homomorphism* for the inclusion $i : X_0 = \{f = 0\} \to X$ ([29, 7]):

$$i^! : H_*(X) \to H_{*-1}(X_0) \quad \text{and} \quad i^! : G_0(X) \to G_0(X_0) \,.$$

Then Verdier's specialization result can be formulated as the equality of the following two transformations:

(3)
$$F(X) \xrightarrow[i^! c_*]{c_* \circ \Psi_f =} H_*(X_0) \,.$$

He also proved this result in the *complex analytic context for compact spaces*. Another proof in the *algebraic context over a base field k of characteristic zero* was later given in [11].

Consider now a base field $k \subset \mathbb{C}$, with $f : X \to \mathbb{A}_k^1$ a morphism and $X_0 := \{f = 0\}$ the fiber over 0. Then one can consider the nearby- and vanishing cycle functors Ψ_f and Φ_f either on the motivic level of localized relative Grothendieck groups

$$\mathcal{M}(var/-) = K_0(var/-)[\mathbb{L}^{-1}]$$

(see [3, 8]), or on the Hodge theoretical level of algebraic mixed Hodge modules over k ([18, 19, 22]), "lifting" under the base change $k \to \mathbb{C}$ the

corresponding functors on the level of algebraically constructible sheaves ([26]) and functions ([25, 29]), so that the following diagram commutes:

$$
(4) \quad
\begin{array}{ccc}
K_0(var/X) & & \\
\downarrow & & \\
\mathcal{M}(var/X) & \xrightarrow{\Psi_f^m, \Phi_f^m} & \mathcal{M}(var/X_0) \\
\chi_{Hdg} \downarrow & & \chi_{Hdg} \downarrow \\
K_0(MHM(X/k)) & \xrightarrow{\Psi_f'^H, \Phi_f'^H} & K_0(MHM(X_0/k)) \\
rat \downarrow & & rat \downarrow \\
K_0\left(D_c^b(X(\mathbb{C}))\right) & \xrightarrow{\Psi_f, \Phi_f} & K_0\left(D_c^b(X_0(\mathbb{C}))\right) \\
\chi_{stalk} \downarrow & & \chi_{stalk} \downarrow \\
F(X(\mathbb{C})) & \xrightarrow{\Psi_f, \Phi_f} & F(X_0(\mathbb{C}))\,.
\end{array}
$$

Here we use the notation $\Psi_f'^H := \Psi_f^H[1]$ and $\Phi_f'^H := \Phi_f^H[1]$ for the shifted functors, with

$$
\Psi_f^H, \Phi_f^H : MHM(X/k) \to MHM(X_0/k)
$$

and

$$
\Psi_f[-1], \Phi_f[-1] : Perv(X(\mathbb{C})) \to Perv(X_0(\mathbb{C}))
$$

preserving mixed Hodge modules and perverse sheaves, respectively. On the level of Grothendieck groups of mixed Hodge modules one has the relation

$$
(5) \qquad \Psi_f'^H = i^* + \Phi_f'^H : K_0(MHM(X/k) \to K_0(MHM(X_0/k))\,,
$$

and similarly for the other transformations. So the vanishing cycles measure the difference between the pullback i^* and the nearby cycles. In particular $[\Psi_f'^H(\mathcal{M}] = [i^*\mathcal{M}]$ in case $\Phi_f'^H(\mathcal{M}) = 0$.

Remark 1.1. The motivic nearby and vanishing cycles functors of [3, 8] take values in a refined *equivariant* localized Grothendieck group $\mathcal{M}^{\hat{\mu}}(var/X)$ of equivariant algebraic varieties over X with a "good" action of the pro-finite group $\hat{\mu} = \lim \mu_n$ of *roots of unity.* In our applications above we don't need to take this action into account.

Also note that for the commutativity of diagram (4) one has to use $\Psi_f'^H$ (as opposed to Ψ_f^H, as stated in [8][Prop.3.17]; this fits in fact with the reference given in the proof of loc.cit). Moreover, the Grothendieck group $\mathcal{M}^{\hat{\mu}}(var/X)$ used in [8] is finer than the one used in [3]. But both definitions of the motivic nearby and vanishing cycle functors are compatible ([8][Rem.3.13]), and χ_{Hdg} also factorizes over $\mathcal{M}^{\hat{\mu}}(var/X)$ in sense of [3] by the same argument as for [8][(3.16.2)].

Before formulating our main result, let us explain a motivating example in the case $k = \mathbb{C}$, with $i : M_0 = \{f = 0\} \to M$ a codimension one inclusion of complex algebraic manifolds (with $m := dim(M)$) and

$$\mathcal{M} \in MHM(M)[m] \subset D^b MHM(M)$$

corresponding to a *"good" variation of mixed Hodge structures* on M, i.e. such that $rat(\mathcal{M})$ is a local system. Then by [27] there is a cohomological characteristic class $MHC^y(\mathcal{M}) \in K^0(M)[y, y^{-1}]$ commuting with the pullback i^*, with $K^0(M)$ the Grothendieck group of algebraic vector bundles or coherent locally free sheaves (of sections), such that

$$MHC_y([\mathcal{M}]) = MHC^y(\mathcal{M}) \cap \left(\Lambda_y([\Omega_M^1]) \cap [\mathcal{O}_M] \right) .$$

Here the pairing

$$\cap = \otimes : K^0(M)[y, y^{-1}] \times G_0(M)[y, y^{-1}] \to G_0(M)[y, y^{-1}]$$

is induced from the tensor product, with

$$\Lambda_y([T^*M]) := \Lambda_y([\Omega_M^1]) := \sum_{i \geq 0} [\Omega_M^i] \cdot y^i \in K^0(M)[y]$$

the *total Lambda class* of the cotangent bundle T^*M. Then one gets

$$
\begin{aligned}
i^! MHC_y([\mathcal{M}]) &= i^* \left(MHC^y(\mathcal{M}) \cup \Lambda_y([\Omega_M^1]) \right) \cap i^!([\mathcal{O}_M]) \\
&= (MHC^y(i^*\mathcal{M}) \cup \Lambda_y([i^*T^*M])) \cap [\mathcal{O}_{M_0}] \\
&= \Lambda_y([N_{M_0}^*M]) \cap MHC_y([i^*\mathcal{M}]) \\
&= (1 + y) \cdot MHC_y(i^*[\mathcal{M}]) .
\end{aligned}
$$
(6)

Here one uses the *multiplicativity* of $\Lambda_y(-)$ with respect to the short exact sequence of vector bundles

$$0 \to N_{M_0}^*M \to T^*M|_{M_0} \to T^*M_0 \to 0 ,$$

and the triviality of the conormal bundle $N_{M_0}^*M$ (coming from the section df) so that

$$\Lambda_y(N_{M_0}^*M) = (1 + y) \cdot [\mathcal{O}_{M_0}] \in K^0(M_0)[y] .$$

But in this special case there are no vanishing cycles $\Phi_f'^H(\mathcal{M}) = 0$ so that

$$MHC_y(i^*[\mathcal{M}]) = MHC_y \left(\Psi_f'^H([\mathcal{M}]) \right)$$

and

$$i^! MHC_y([\mathcal{M}]) = (1 + y) \cdot MHC_y \left(\Psi_f'^H([\mathcal{M}]) \right) .$$

And this formula holds in general, i.e. assume now that $X_0 = \{f = 0\}$ is a hypersurface of codimension one in the algebraic variety X over $k \subset \mathbb{C}$. Then we prove the following counterpart of Verdier's specialization result:

Theorem 1.1. *The motivic Hodge-Chern class transformation MHC_y commutes with specialization in the following sense:*

(7) $(1+y) \cdot MHC_y(\, \Psi_f'^H(-)\,) = -(1+y) \cdot MHC_y(\, \Psi_f^H(-)\,) = i^! MHC_y(-)$

as transformations $K_0(MHM(X/k)) \to G_0(X_0)[y, y^{-1}]$.

Remark 1.2. Our proof also gives the same result in the *embedded complex analytic* context, with $X \subset M$ a complex analytic subspace of the complex manifold M and $f : M \to \mathbb{C}$ a holomorphic function such that $X_0 := X \cap \{f = 0\}$ is a *compact* hypersurface of codimesion one in X. Here we use analytic mixed Hodge modules $MHM(X)$ on X ([18, 19]), which we identify with analytic mixed Hodge modules $MHM_X(M)$ on M with support in X (and similarly for X_0). Here $G_0(X)$ is of course the Grothendieck group of analytic coherent $\mathcal{O}_X$-sheaves, which has a canonical map to the Grothendieck group $G_0^X(M)$ of analytic coherent $\mathcal{O}_M$-sheaves on M with support in X. And $G_0(X_0) = G_0^{X_0}(M)$, since X_0 is compact.

Let us come back to the algebraic context. Then another earlier result of Verdier [28, 7] states that the Todd class transformation of Baum-Fulton-MacPherson [1, 7]

$$td_* : G_0(-) \to H_*(-) \otimes \mathbb{Q}$$

commutes with the Gysin homomorphisms $i^!$ in these homology theories. The *unnormalized* Hirzebruch class transformation $MH\tilde{T}_{y*}$ of [4] is defined as

$$MH\tilde{T}_{y*} := td_* \circ MHC_y : K_0(MHM(-/k)) \to H_*(-) \otimes \mathbb{Q}[y, y^{-1}] \,.$$

Since td_* commutes with specialization, we get the

Corollary 1.1. *Consider an algebraic function* $f : X \to \mathbb{A}_k^1$ *such that the inclusion* $i : X_0 := \{f = 0\} \to X$ *of the zero fiber is everywhere of codimension one. Then the Hirzebruch class transformations* $MH\tilde{T}_{y*}$ *and* MHT_{y*} *commute with specialization in the following sense:*

(8) $$(1+y) \cdot MH\tilde{T}_{y*}(\Psi_f'^H(-)) = i^! MH\tilde{T}_{y*}(-)$$

as transformations $K_0(MHM(X/k)) \to H_*(X_0) \otimes \mathbb{Q}[y, y^{-1}]$.

(9) $$MHT_{y*}(\Psi_f'^H(-)) = i^! MHT_{y*}(-)$$

as transformations $K_0(MHM(X/k)) \to H_*(X_0) \otimes \mathbb{Q}[y, y^{-1}, (1+y)^{-1}]$. □

By the definition of Ψ_f^m in [3, 8] one has

$$\Psi_f^m(K_0(var/M)) \subset im(K_0(var/X) \to \mathcal{M}(X)) \,,$$

so that

$$MHT_{y*} \circ \Psi_f^m : K_0(var/M) \to H_*(X) \otimes \mathbb{Q}[y] \subset H_*(X) \otimes \mathbb{Q}[y, y^{-1}] \,.$$

Together with [27][Prop.5.2.1] one therefore gets the following commutative diagram of specialization results, where one uses the base change $k \to \mathbb{C}$ for

the last part:

$$
(10) \quad
\begin{array}{ccc}
K_0(var(X)) & \xrightarrow[\;i^! \circ T_{y_*}\;]{\;T_{y_*} \circ \Psi_f^m =\;} & H_*(X_0) \otimes \mathbb{Q}[y] \\[4pt]
\;\;\downarrow{\scriptstyle \chi_{Hdg}} & & \downarrow \\[6pt]
K_0(MHM(X/k)) & \xrightarrow[\;i^! \circ MHT_{y_*}\;]{\;MHT_{y_*} \circ \Psi_f'^H =\;} & H_*(X_0) \otimes \mathbb{Q}[y, y^{-1}] \\[4pt]
\;\;\downarrow{\scriptstyle \chi_{stalk} \circ rat} & & \downarrow{\scriptstyle y=-1} \\[6pt]
F(X(\mathbb{C})) & \xrightarrow[\;i^! c_*\;]{\;c_* \circ \Psi_f =\;} & H_*(X_0(\mathbb{C})) \otimes \mathbb{Q} \,.
\end{array}
$$

Applications of this corollary to the calculation of the Hirzebruch class

$$T_{y*}(X) := T_{y*}([id_X])$$

of a global hypersurface $X := \{f = 0\}$ inside a manifold M in the complex algebraic context are given in our recent joint paper [5].

Note that the motivic Chern class transformation MHC_y is defined in terms of the *filtered de Rham complex* of the filtered $\mathcal{D}$-module underlying a mixed Hodge module ([4, 27]). So in the embedded context $X \subset M$ a closed subspace of the manifold M, with $f : M \to \mathbb{A}_k^1$, our main result theorem 1.1 becomes a purely $\mathcal{D}$-module theoretic result about coherent $\mathcal{D}_M$-modules with a good (Hodge) filtration F supported on X, which are *strictly specializable* with respect to the hypersurface $\{f = 0\}$. Here one uses the $\mathcal{D}$-module description of nearby cycles in terms of the *V-filtration of Malgrange-Kashiwara* ([14, 9, 15]). Then the strict specializability is a condition on the relation between the Hodge filtration F and the V-filtration introduced by M. Saito [18] in his definition of *pure Hodge modules*. So the filtered $\mathcal{D}$-module underlying a pure (and also a mixed) Hodge module satisfies this technical condition (more or less) by definition. And in the next section we will show that in the embedded context exactly this *strict specializability* is needed for the proof of our main result! Using *resolution of singularities and functoriality* under proper morphisms, we deduce in the last section our main result for mixed Hodge modules on a singular space X from the corresponding $\mathcal{D}$-module result in the embedded context.

2. Specialization for $\mathcal{D}$-modules

We are working either in the algebraic context over a base field $k \subset \mathbb{C}$, i.e. all sheaves are algebraic sheaves in the Zariski topology, or in the complex analytic context. Consider an algebraic (or analytic) manifold M of dimension n, together with

$$t = pr_2 : M' := M \times \mathbb{A}_k^1 \to \mathbb{A}_k^1$$

the projection onto the second factor. Of course in the complex analytic context, we consider the affine line $\mathbb{C} = \mathbb{A}_C^1$ just as a complex manifold. Let $i': M \simeq M \times \{0\} \to M' := M \times \mathbb{C}$ be the closed inclusion of the zero fiber.

Let $\mathcal{D}_{M'}$ be the corresponding coherent sheaf of algebraic or analytic differential operators. This has an increasing filtration F by locally free coherent $\mathcal{O}_{M'}$-sheaves $F^i\mathcal{D}_{M'}$ ($i \in \mathbb{N}_0$) given by the differential operators of order less or equal i, with

$$\cup_{i \geq 0} F^i\mathcal{D}_{M'} = \mathcal{D}_{M'} .$$

(see e.g. [15, 23]). Let $\mathcal{I} \subset \mathcal{O}_{M'}$ be the ideal sheaf defining M, i.e. the sheaf of functions vanishing along M. Then the *increasing V-filtration of Malgrange-Kashiwara* with respect to the smooth hypersurface $M \subset M'$ is for $k \in \mathbb{Z}$ *locally* defined by

$$V_k\mathcal{D}_{M'} := \{P \in \mathcal{D}_{M'} \mid P(\mathcal{I}^{j+k}) \subset \mathcal{I}^j \quad \text{for all } j \in \mathbb{Z}\} .$$

Here $\mathcal{I}^j := \mathcal{O}_{M'}$ for $j < 0$. Note that

$$\cap_{k \in \mathbb{Z}} V_k\mathcal{D}_{M'} = \{0\} \quad \text{and} \quad \cup_{k \in \mathbb{Z}} V_k\mathcal{D}_{M'} = \mathcal{D}_{M'} .$$

Moreover $V_k\mathcal{D}_{M'}|\{f \neq 0\} = \mathcal{D}_{M'}|\{f \neq 0\}$ for all $k \in \mathbb{Z}$ so that $gr_k^V\mathcal{D}_{M'}$ is supported on M. By definition one has

$$t \in V_{-1}\mathcal{D}_{M'}, \ \partial_t \in V_1\mathcal{D}_{M'} \quad \text{and} \quad \partial_t t = 1 + t\partial_t \in V_0\mathcal{D}_{M'} .$$

Similarly we have for the sheaf $\mathcal{D}_{M'/A}$ of *relative differential operators along the fibers of t*:

$$\mathcal{D}_{M'/A} \subset V_0\mathcal{D}_{M'} \quad \text{and} \quad gr_0^V\mathcal{D}_{M'}|M = \mathcal{D}_M[\partial_t t] .$$

Definition 2.1 (Specializability)**.** *A coherent left $\mathcal{D}_{M'}$-module $\mathcal{M}$ is said to be* specializable *along M if for any (closed) point $x \in M$ (with residue field k_x) and any local section $m \in \mathcal{M}_x$ there is a nonzero polynomial $b(s) \in k_x[s]$ such that*

(11) $$b(\partial_t t) \cdot m \in V_{-1}\mathcal{D}_{M',x} \cdot m .$$

The Bernstein polynomial (or b-function) b_m *of $m \in \mathcal{M}_x$ is the monic polynomial in $k_x[s]$ of minimal degree satisfying the condition (11).*

For example any *holonomic* $\mathcal{D}_{M'}$-module is specializable along M (compare e.g. [15][prop.4.4.2]). Similarly, for a short exact sequence

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$$

of coherent $\mathcal{D}_{M'}$-modules one has $\mathcal{M}$ is specializable if and only if $\mathcal{M}'$ and $\mathcal{M}''$ are specializable ([15][prop.4.2.4]).

Let us fix a total order $\leq$ on $\mathbb{C}$ with the following properties for all $\alpha, \beta \in \mathbb{C}$ and $a \in \mathbb{R}$:
- (o1) it induces the usual order on $\mathbb{R}$,
- (o2) $\alpha + a < \beta + a \Leftrightarrow \alpha < \beta$,

(o3) $\alpha < \alpha + 1$ and there is some $m \in \mathbb{N}$ with $\alpha < \beta + m$.

Definition 2.2 (Canonical V-filtration)**.** *Let* $\mathcal{M}$ *be a coherent* left $\mathcal{D}_{M'}$-*module, which is specializable with respect to the hypersurface* M. *Then the canonical increasing* V-*filtration of* $\mathcal{M}$ *(indexed by* $\mathbb{C}$ *and* $<$*) is defined as*

$$(12) \quad V_\alpha \mathcal{M}_x := \{ m \in \mathcal{M}_x | \text{ all roots of the b-function } b_m \text{ are } \geq -\alpha - 1 \}$$

for all (closed) points $x \in M$ *(compare* [15]*[def.4.3.3]). In the complex analytic context we assume in addition that* $M \cap supp(\mathcal{M})$ *is* compact. *Then the canonical* V-*filtration is indexed* discretely *by* $A + \mathbb{Z} \subset \bar{k} \subset \mathbb{C}$ *for a finite subset* $A \subset \bar{k} \subset \mathbb{C}$, *and it is the* unique *such filtration* V *with the following properties:*

(1) $\bigcup_\alpha V_\alpha \mathcal{M} = \mathcal{M}$, *and each* $V_\alpha \mathcal{M}$ *is a coherent* $V_0 \mathcal{D}_{M'}$ *module.*
(2) $(V_k \mathcal{D}_{M'})(V_\alpha \mathcal{M}) \subset V_{\alpha+k} \mathcal{M}$ *for all* $\alpha \in \mathbb{C}, k \in \mathbb{Z}$,
   and $t(V_\alpha \mathcal{M}) = V_{\alpha-1} \mathcal{M}$ *for all* $\alpha < 0$.
(3) $\partial_t t + \alpha$ *is* nilpotent *on* $gr_\alpha^V \mathcal{M} := V_\alpha \mathcal{M} / V_{<\alpha} \mathcal{M}$,
   with $V_{<\alpha} \mathcal{M} := V_\beta \mathcal{M}$ *for* $\beta := \sup\{\beta' \in A + \mathbb{Z} | \beta' < \alpha\}$.

These conditions imply $t : gr_\alpha^V \mathcal{M} \to gr_{\alpha-1}^V \mathcal{M}$ is bijective for all $\alpha \neq 0$, and $\partial_t : gr_\alpha^V \mathcal{M} \to gr_{\alpha+1}^V \mathcal{M}$ is bijective for all $\alpha \neq -1$. Moreover (compare [15][prop.4.4.3] and [18][lem.3.1.4])

$$(13) \qquad t \cdot : V_\alpha \mathcal{M} \to V_{\alpha-1} \mathcal{M} \quad \text{is injective for all } \alpha < 0.$$

Finally all $gr_\alpha^V \mathcal{M} | M$ are coherent *left* $\mathcal{D}_M$-modules ([15][cor.4.3.10]), which are also all (regular) holonomic, if $\mathcal{M}$ has this property ([15][cor.4.6.3, cor.4.7.5]). In the algebraic context over a base field k this means by definition that the corresponding $\mathcal{D}$-modules over $M'(\mathbb{C})$ and $M(\mathbb{C})$ obtained by the base extension $k \to \mathbb{C}$ are (regular) holonomic.

Definition 2.3. *Let* $\mathcal{M}$ *be a coherent* left $\mathcal{D}_{M'}$-*module, which is specializable with respect to the hypersurface* M. *Then* $\mathcal{M}$ *is called* quasi-unipotent *(along* M*), if for all* $m \in \mathcal{M}_x$ *with* x *a (closed) point of* M *the Berstein polynomial* $b_m(s) \in k_x[s]$ *has only rational roots in* $\mathbb{Q} \subset k \subset \bar{k}$. *Or in other words, if the canonical* V-*filtration is discretely indexed by* $\mathbb{Q}$ *(so that the total order* $<$ *on* $\mathbb{C}$ *above isn't needed).*

Remark 2.1. Here we are working with an *increasing* V-filtration for *left* $\mathcal{D}$-modules, so that $\pm(\partial_t t + \alpha)$ is *nilpotent* on $gr_\alpha^V \mathcal{M}$.

(1) If we work with the associated *analytic* $\mathcal{D}$-module $\mathcal{M}^{an}$, then $(V_\alpha \mathcal{M})^{an}$ is the corresponding V-filtration in the analytic context (by uniqueness). Similarly in the algebraic context over a base field $k \subset \mathbb{C}$, if one takes the base extension $k \to \bar{k}$ or $k \to \mathbb{C}$.
(2) If we switch to the corresponding *right* $\mathcal{D}$-module $\mathcal{M}^r := \omega_{M'} \otimes \mathcal{M}$, with the induced V-filtration $\omega_{M'} \otimes V_\alpha \mathcal{M}$, then $t\partial_t - \alpha$ is nilpotent on $gr_\alpha^V \mathcal{M}^r$ (fitting with the convention of [18][Def. 3.1.1]).

(3) If we use the corresponding *decreasing V-filtration* $V^\alpha := V_{-\alpha-1}$ for *left $\mathcal{D}$-modules*, then $t\partial_t - \alpha$ is nilpotent on $gr_V^\alpha \mathcal{M}$ (fitting with the convention of [18][Introduction, p.851]).

Assume now that $\mathcal{M}$ in addition is endowed with a *good filtration F*, i.e. with an increasing filtration $F_p\mathcal{M}$ ($p \in \mathbb{Z}$) by coherent $\mathcal{O}_{M'}$-modules, such that

(1) $F_p\mathcal{M} = 0$ for $p << 0$ small enough,
(2) $\cup_{p\in\mathbb{Z}} F_p\mathcal{M} = \mathcal{M}$,
(3) $(F_k\mathcal{D}_M)(F_p\mathcal{M}) \subset F_{k+p}\mathcal{M}$ for all $k, p \in \mathbb{Z}$,
(4) $gr^F\mathcal{M}$ is a coherent $gr^F\mathcal{D}_{M'}$ module.

Then one can ask about the relation between the two filtrations F and V. The following notion will be important for our main results (compare [18][Def.3.2.1, p.905] and [17][conditions 4.3.(b-c), p.46]):

Definition 2.4 (Strict specializability). *Let $\mathcal{M}$ be a coherent left $\mathcal{D}_{M'}$-module, which is specializable with respect to the hypersurface M. Assume that it is also endowed with a good filtration F. The filtered $\mathcal{D}_{M'}$-module $(\mathcal{M}, F)$ is called* strictly specializable *along M, iff*

s1. *The induced F-filtration on $gr_\alpha^V\mathcal{M}$ is good for all $-1 \leq \alpha \leq 0$.*
s2. *$t(F_pV_\alpha\mathcal{M}) = F_pV_{\alpha-1}\mathcal{M}$ for all $\alpha < 0$ and $p \in \mathbb{Z}$.*
s3. *$\partial_t(F_p gr_\alpha^V\mathcal{M}) = F_{p+1}gr_{\alpha+1}^V\mathcal{M}$ for all $\alpha > -1$ and $p \in \mathbb{Z}$.*

Here the *induced F-filtration* on $V_\alpha\mathcal{M}$ resp. $gr_\alpha^V\mathcal{M}$ are given by (cf. [6])

$$F_pV_\alpha\mathcal{M} := F_p\mathcal{M} \cap V_\alpha\mathcal{M}$$

resp.

$$F_p gr_\alpha^V\mathcal{M} := im\left(F_pV_\alpha\mathcal{M} \to gr_\alpha^V\mathcal{M}\right) \simeq F_pV_\alpha\mathcal{M}/F_pV_{<\alpha}\mathcal{M}.$$

Then the induced short exact sequence of filtered $\mathcal{O}_{M'}$-modules

(14) $$0 \to (V_{<\alpha}\mathcal{M}, F) \to (V_\alpha\mathcal{M}, F) \to (gr_\alpha^V\mathcal{M}, F) \to 0$$

is *strict* (compare [6, 12]) in the sense that

$$0 \to F_pV_{<\alpha}\mathcal{M} \to F_pV_\alpha\mathcal{M} \to F_p gr_\alpha^V\mathcal{M} \to 0$$

is exact for all $p \in \mathbb{Z}$, which also implies that

$$0 \to gr_p^F V_{<\alpha}\mathcal{M} \to gr_p^F V_\alpha\mathcal{M} \to gr_p^F gr_\alpha^V\mathcal{M} \to 0$$

is exact for all $p \in \mathbb{Z}$. Finally one has a *canonical isomorphism* (compare [6][(1.2.1)]):

(15) $$gr_p^F gr_\alpha^V\mathcal{M} \simeq gr_\alpha^V gr_p^F\mathcal{M} \quad \text{for all } p \in \mathbb{Z}, \alpha \in \mathbb{C}.$$

And similarly for V_α/V_β instead of gr_α^V for $\alpha < \beta$.

Remark 2.2. By Saito's work [18, 19, 22], the underlying filtered D-module of an *algebraic or analytic pure Hodge module* on M' is *strictly specializable and quasi-unipotent* along M.

(1) In the analytic context this is so by definition (where we again assume that the intersection of the support with M is compact, so that the canonical V-filtration is indexed discretely).

(2) In the complex algebraic context a pure Hodge module of [19] is by definition "extendable and quasi-unipotent" at infinity so that one can assume M is compact and then the properties (s1-s3) in the analytic context of [18] imply by GAGA and flatness of $\mathcal{O}^{an}$ over $\mathcal{O}$ the same properties for the underlying algebraic filtrations.

(3) In the algebraic context over a base field $k \subset \mathbb{C}$ the claim follows from the complex algebraic context by the exact and flat base change $k \to \mathbb{C}$ (compare [22][rem. on p.9]).

(4) Then also the underlying filtered D-module of an *algebraic or analytic mixed Hodge module* on M' as in [19, 22] is *strictly specializable and quasi-unipotent* along M, since it is a finite successive extension of pure Hodge modules (by the weight filtration). And the corresponding short exact extension sequences of mixed Hodge modules a strict with respect to the (Hodge) filtration F. Since also the canonical V-filtration behaves well under extensions, one easily gets the quasi-unipotence and the condition (s1-s3) above by induction (using the properties of the underlying V-filtrations from definition 2.2).

For a closed algebraic (or analytic) subset $X' \subset M'$ we get a cartesian diagram of inclusions:

$$
\begin{array}{ccc}
X := X \cap \{t = 0\} & \xrightarrow{\ i\ } & X' \\
\downarrow & & \downarrow \\
M & \xrightarrow[\ i'\ ]{} & M' \,.
\end{array}
$$

And we are interested in the algebraic (or analytic) *de Rham complex* $DR^*(\mathcal{M})$ of a filtered $\mathcal{D}_{M'}$-module $(\mathcal{M}, F)$ supported on X':

$$(16) \qquad \left[\cdots \Omega^k_{M'} \otimes \mathcal{M} \xrightarrow{\ \nabla^k\ } \Omega^{k+1}_{M'} \otimes \mathcal{M} \cdots \right],$$

with $\Omega^{n+1}_{M'} \otimes \mathcal{M}$ put in *degree zero* (and $n = dim(M)$, $n + 1 = dim(M')$). Here ∇^k comes from the *integrable connection* ∇ on $\mathcal{M}$ given by the *left* $\mathcal{D}_{M'}$-module structure:

$$(17) \qquad \nabla^k(\omega \otimes m) = d\omega \otimes m + (-1)^k \omega \wedge \nabla m \,.$$

From this it follows that the de Rham complex gets an induced filtration

$$(18) \quad F_p DR^*(\mathcal{M}) = \left[\cdots \Omega^k_{M'} \otimes F_{p+k}\mathcal{M} \xrightarrow{\ \nabla^k\ } \Omega^{k+1}_{M'} \otimes F_{p+k+1}\mathcal{M} \cdots \right],$$

such that differentials $gr^F_p(\nabla^k)$ of the graded complex

$$(19) \quad gr^F_p DR^*(\mathcal{M}) = \left[\cdots \Omega^k_{M'} \otimes gr^F_{p+k}\mathcal{M} \xrightarrow{\ gr^F_p(\nabla^k)\ } \Omega^{k+1}_{M'} \otimes gr^F_{p+k+1}\mathcal{M} \cdots \right]$$

are $\mathcal{O}_{M'}$-linear! Since the filtration F of $\mathcal{M}$ is *good*, one also knows (compare [18][Lem.2.1.17, p.882]) that

$$(20) \qquad\qquad gr_p^F DR^*(\mathcal{M}) \in D_{coh,X'}^b(M')$$

is a bounded complex with coherent cohomology supported on X', which is acyclic for almost all p.

Definition 2.5. *The* motivic Hodge-Chern class $MHC_y((\mathcal{M}, F))$ *of the coherent $\mathcal{D}$-module $(\mathcal{M}, F)$ with its good filtration F is given by*

$$(21) \quad MHC_y(\,(\mathcal{M}, F)\,) := \sum_p [gr_{-p}^F DR^*(\mathcal{M})] \cdot (-y)^p \in G_0(X')[y, y^{-1}]\,.$$

Here we first identify the derived Grothendieck group

$$K_0\left(D_{coh,X'}^b(M')\right) = G_0^{X'}(M')$$

with the Grothendieck group $G_0^{X'}(M')$ of coherent $\mathcal{O}_{M'}$-sheaves with support in X', and the class $[-]$ of a complex means the alternating sum of the classes of its cohomology sheaves. Next the closed inlusion $\kappa : X' \to M'$ induces a natural group homomorphism on the Grothendieck group $G_0(X')$ of coherent $\mathcal{O}_{X'}$-sheaves:

$$\kappa_* : G_0(X') \to G_0^{X'}(M')\,.$$

By the Hilbert-Rückert theorem this is an isomorphism in the algebraic context, as well as in the analytic context for X' compact (or relatively compact, if we are "allowed to take a shrinking").

Remark 2.3. If $(\mathcal{M}, F)$ underlies a mixed Hodge module with support in X', then all $gr_p^F \mathcal{M}$ are in fact $\mathcal{O}_{X'}$-modules so that $gr_p^F DR^*(\mathcal{M})$ is already a well defined complex in $D_{coh}^b(X')$! Just take locally a function on M' vanishing along X'. Then $(\mathcal{M}, F)$ is also "strictly specializable along g", which implies

$$g(F_p\mathcal{M}) \subset F_{p-1}\mathcal{M} \quad \text{and} \quad g(gr_p^F \mathcal{M}) = 0$$

by [18][lem.3.2.6].

Now we want to compare the specialization (or nearby cycles) $\Psi_t((\mathcal{M}, F))$ of the *strictly specializable* pair $(\mathcal{M}, F)$ with the specialization $i^!$ of the motivic Hodge-Chern classes, where the Gysin homomorphism

$$i^! : G_0(X') \to G_0(X)$$

is induced by the derived pullback map

$$Li^* : D_{coh,X'}^b(M') \to D_{coh,X}^b(M)\,.$$

By the short exact sequence

$$0 \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_{M'} \xrightarrow{\;t\cdot\;} \mathcal{O}_{M'} \longrightarrow 0\,,$$

this Gysin homomorphism $i^!$ is also represented by taking the (class of the) tensor product with the complex $[t \cdot : \mathcal{O}_{M'} \to \mathcal{O}_{M'}]$. In particular

$$(22) \qquad i^! i_* = 0 : G_0(X) \to G_0(X); \ [\mathcal{F}] \mapsto [\mathcal{F} \xrightarrow{t \cdot} \mathcal{F}] \, .$$

Let us now recall the

Definition 2.6. *The* nearby cycles $\Psi_t((\mathcal{M}, F))$ *of the strictly specializable pair* $(\mathcal{M}, F)$ *are given by*

$$(23) \qquad \Psi_t((\mathcal{M}, F)) := \sum_{-1 \leq \alpha < 0} \Psi_{t,\alpha}((\mathcal{M}, F)) \, ,$$

with $\Psi_{t,\alpha}((\mathcal{M}, F)) := (gr_\alpha^V \mathcal{M} | M, F)$.
The unipotent vanishing cycles $\Phi_{t,uni}((\mathcal{M}, F))$ *of* $(\mathcal{M}, F)$ *are given by*

$$(24) \qquad \Phi_{t,uni}((\mathcal{M}, F)) := (gr_0^V \mathcal{M} | M, F[-1]) \, ,$$

with the shifted filtration defined as $(F[k])_i := F_{i-k}$.

Remark 2.4. Our defintion of the induced filtration fits with [18][intro., p.851], since we are using *left* $\mathcal{D}$-modules. For the corresponding *right* $\mathcal{D}$-modules one has to shift these induced F-filtrations by [+1] (compare [18][(5.1.3.3) on p.953]). This corrects then the different switching by

$$\otimes(\omega_{M'}, F) \quad \text{or} \quad \otimes(\omega_M, F)$$

from filtered left $\mathcal{D}$-modules to filtered right $\mathcal{D}$-modules on M' or M, with F the trivial filtration such that $gr_{-k}^F(-) = 0$ for $k \neq$ the dimension of the ambient manifold M' or M.

Now we can formulate our main result of this section.

Theorem 2.1. *Let* $(\mathcal{M}, F)$ *be a coherent* $\mathcal{D}_{M'}$*-module* $\mathcal{M}$ *supported on* X' *endowed with a good filtration* F, *which is strictly specializable along* M. *Then the motivic Hodge-Chern classes commute with specialization in the following sense:*

$$(25) \qquad -(1+y) \cdot MHC_y(\Psi_t((\mathcal{M}, F))) = i^! MHC_y((\mathcal{M}, F))$$

as classes in $G_0(X)[y, y^{-1}]$.

The proof is given on the next pages. It uses of course the properties (s1.)-(s3.). From (s3.) one gets the

Lemma 2.1.

$$\partial_t : (\mathcal{M}/V_\alpha \mathcal{M}, F_*) \to (\mathcal{M}/V_{\alpha+1} \mathcal{M}, F_{*+1})$$

is a filtered isomorphism *for all* $\alpha \geq -1$, *so that also*

$$\partial_t : gr_p^F(\mathcal{M}/V_\alpha \mathcal{M}) \to gr_{p+1}^F(\mathcal{M}/V_{\alpha+1} \mathcal{M})$$

is an isomorphism *for all* $\alpha \geq -1$ *and* $p \in \mathbb{Z}$. $\qquad\qquad \square$

This follows by $\bigcup_\alpha V_\alpha \mathcal{M} = \mathcal{M}$ from the corresponding result:

$$\partial_t : (V_\beta/V_\alpha \mathcal{M}, F_*) \to (V_{\beta+1}/V_{\alpha+1}\mathcal{M}, F_{*+1})$$

is a *filtered isomorphism* for all $\beta > \alpha \geq -1$.

And this follows by induction from (s3.) and the short exact sequence

$$
\begin{array}{ccccc}
0 \to (V_\beta/V_\alpha \mathcal{M}, F_*) & \longrightarrow & (V_{\beta'}/V_\alpha \mathcal{M}, F_*) & \longrightarrow & \cdots \\
\downarrow \partial_t & & \downarrow \partial_t & & \\
0 \to (V_{\beta+1}/V_{\alpha+1}\mathcal{M}, F_{*+1}) & \longrightarrow & (V_{\beta'+1}/V_{\alpha+1}\mathcal{M}, F_{*+1}) & \longrightarrow & \cdots
\end{array}
$$

$$\cdots (gr^V_{\beta'}\mathcal{M}, F_*) \to 0$$

$$\partial_t \downarrow$$

$$\cdots (gr^V_{\beta'+1}\mathcal{M}, F_{*+1}) \to 0 \,,$$

where $\beta' \in A + \mathbb{Z}$ is smallest number bigger than β. Of course we also use the fact that $\partial_t : gr^V_\beta \mathcal{M} \to gr^V_{\beta+1}\mathcal{M}$ is bijective for all $\beta > -1$.

Similarly one gets from (s2.) and (13) by induction over p (with $F_p(-) = 0$ for $p << 0$) the

Lemma 2.2. *The multiplication*

$$t \cdot : gr^F_p(V_\alpha \mathcal{M}) \to gr^F_p(V_{\alpha-1}\mathcal{M})$$

is an isomorphism *for all $\alpha < 0$ and $p \in \mathbb{Z}$. In particular*

$$Li^* gr^F_p(V_\alpha \mathcal{M}) = i^* gr^F_p(V_\alpha \mathcal{M}) = gr^F_p(\, V_\alpha/V_{\alpha-1}\mathcal{M}|M\,)$$

for all $\alpha < 0$ and $p \in \mathbb{Z}$. $\square$

Now we can come to the main geometric idea. Since we work on the product manifold $M' = M \times \mathbb{A}^1_k$ (which usually comes from a graph embedding for an algebraic (or analytic) function $f : M \to \mathbb{A}^1_k$), we have the splitting

$$\Omega^1_{M'} = \Omega^1_{M'/A} \oplus \Omega^1_A \,,$$

with $\Omega^1_{M'/A}$ the relative 1-forms with respect to the submersion t and $\Omega^1_A \simeq \mathcal{O}_{M'}dt$. So we get for $0 \leq k \leq n + 1$:

$$\Omega^k_{M'} = \Omega^k_{M'/A} \oplus \Omega^1_A \otimes \Omega^{k-1}_{M'/A} \,.$$

And this induces a splitting of the de Rham complex $DR^*(\mathcal{M})$ as a double complex

$$
\begin{array}{ccccc}
DR^*_{/A}(\mathcal{M}) & = & [\cdots \Omega^k_{M'/A} \otimes \mathcal{M} & \xrightarrow{\nabla^k_{/A}} & \Omega^{k+1}_{M'/A} \otimes \mathcal{M} \cdots] \\
\nabla_t \downarrow & & \nabla^k_t \downarrow & & \nabla^{k+1}_t \downarrow \\
\Omega^1_A \otimes DR^*_{/A}(\mathcal{M}) & = & [\cdots \Omega^1_A \otimes \Omega^k_{M'/A} \otimes \mathcal{M} & \xrightarrow{\nabla^k_{/A}} & \Omega^1_A \otimes \Omega^{k+1}_{M'/A} \otimes \mathcal{M} \cdots] \,,
\end{array}
$$

with the "top-dimensional forms" $\Omega^1_A \otimes \Omega^n_{M'/A}$ in bidegree $(0,0)$.

Here the horizontal lines come from the corresponding *relative de Rham complex* $DR^*_{/A}(\mathcal{M})$ of $\mathcal{M}$ viewed only as left $\mathcal{D}_{M'/A}$-module, whereas the vertical maps are given by:

$$\nabla^k_t : \Omega^k_{M'/A} \otimes \mathcal{M} \to \Omega^1_A \otimes \Omega^k_{M'/A} \otimes \mathcal{M};$$

$$\omega \otimes m \mapsto dt \otimes (\partial_t \omega) \otimes m + (-1)^k dt \otimes \omega \otimes (\partial_t m) .$$

In particular $DR^*(\mathcal{M})$ becomes a bifiltered double complex by

$$F_p V_\alpha DR^*_{/A}(\mathcal{M}) \quad\quad\quad ==$$

$$\nabla_t \downarrow$$

$$\Omega^1_C \otimes F_{p+1} V_{\alpha+1} DR^*_{/A}(\mathcal{M}) \quad ==$$

$$[\cdots \Omega^k_{M'/A} \otimes F_{p+k} V_\alpha \mathcal{M} \quad \xrightarrow{\nabla^k_{/A}} \quad \Omega^{k+1}_{M'/A} \otimes F_{p+k+1} V_\alpha \mathcal{M} \cdots]$$

$$\nabla^k_t \downarrow \quad\quad\quad\quad\quad\quad\quad\quad \nabla^{+1}_t k \downarrow$$

$$[\cdots \Omega^1_A \otimes \Omega^k_{M'/A} \otimes F_{p+k+1} V_{\alpha+1} \mathcal{M} \quad \xrightarrow{\nabla^k_{/A}} \quad \Omega^1_A \otimes \Omega^{k+1}_{M'/A} \otimes F_{p+k+2} V_{\alpha+1} \mathcal{M} \cdots] .$$

Again all differentials in the F-graded and V-filtered complex

$$(26) \quad\quad\quad\quad gr^F_p V_\alpha DR^*_{/A}(\mathcal{M})$$

$$gr^F(\nabla_t) \downarrow$$

$$\Omega^1_A \otimes gr^F_{p+1} V_{\alpha+1} DR^*_{/A}(\mathcal{M})$$

are $\mathcal{O}_{M'}$-linear. Moreover, the induced vertical $\mathcal{O}_{M'}$-linear maps

$$gr^F_p(\nabla^k_t) : \Omega^k_{M'/A} \otimes gr^F_p V_\alpha \mathcal{M} \to \Omega^1_A \otimes \Omega^k_{M'/A} \otimes gr^F_{p+1} V_{\alpha+1} \mathcal{M};$$

$$\omega \otimes [m] \mapsto (-1)^k dt \otimes \omega \otimes (\partial_t [m])$$

correspond up to a sign $(-1)^k$ under the identification

$$\Omega^1_A \simeq \mathcal{O}_{M'}; \; f dt \simeq f$$

to the maps induced from

$$\partial_t : (V_\alpha \mathcal{M}, F_p) \to (V_{\alpha+1} \mathcal{M}, F_{p+1})$$

by tensoring with the free (and therefore flat) $\mathcal{O}_{M'}$-modules $\Omega^k_{M'/A}$!

So by Lemma 0.1 we get the

Corollary 2.1. *The horizontal inclusion of F-filtered double complexes*

$$
\begin{array}{ccc}
F_p V_{-1} DR^*_{/A}(\mathcal{M}) & \longrightarrow & F_p DR^*_{/A}(\mathcal{M}) \\
\nabla_t \downarrow & & \nabla_t \downarrow \\
\Omega^1_A \otimes F_{p+1} V_0 DR^*_{/A}(\mathcal{M}) & \longrightarrow & \Omega^1_A \otimes F_{p+1} DR^*_{/A}(\mathcal{M})
\end{array}
$$

induces a filtered quasi-isomorphism of the correspondung total complexes, i.e. the horizontal map

$$
\begin{array}{ccc}
gr^F_p V_{-1} DR^*_{/A}(\mathcal{M}) & \longrightarrow & gr^F_p DR^*_{/A}(\mathcal{M}) \\
gr^F_p(\nabla_t) \downarrow & & gr^F_p(\nabla_t) \downarrow \\
\Omega^1_A \otimes gr^F_{p+1} V_0 DR^*_{/A}(\mathcal{M}) & \longrightarrow & \Omega^1_A \otimes gr^F_{p+1} DR^*_{/A}(\mathcal{M})
\end{array}
$$

induces a quasi-isomorphism of the correspondung total complexes for all $p \in \mathbb{Z}$. $\hfill\square$

In fact all vertical maps of the corresponding quotient double complex

$$
gr^F_p DR^*_{/A}(\mathcal{M}/V_{-1}\mathcal{M})
$$
$$
gr^F_p(\nabla_t) \downarrow
$$
$$
\Omega^1_A \otimes gr^F_{p+1} DR^*_{/A}(\mathcal{M}/V_0\mathcal{M})
$$

are isomorphisms by Lemma 0.1 so that the total complex of this quotient double complex is *acyclic*! Note that the filtered relative de Rham complex

$$
\left(DR^*_{/A}(-), F \right)
$$

is *functorial* for filtered $(\mathcal{D}_{M'/A}, F)$-modules, where the sheaf of relative differential operators $\mathcal{D}_{M'/A} \subset V_0 \mathcal{D}_{M'}$ is again filtered by the order of a differential operator.

So the total complex of

(27)
$$
gr^F_p V_{-1} DR^*_{/A}(\mathcal{M})
$$
$$
gr^F_p(\nabla_t) \downarrow
$$
$$
\Omega^1_A \otimes gr^F_{p+1} V_0 DR^*_{/A}(\mathcal{M})
$$

represents $gr^F_p DR^*(\mathcal{M})$. Therefore it belongs to $D^b_{coh,X'}(M')$, is acyclic for almost all p, and can be used for the calculation of

$$
MHC_y((\mathcal{M}, F)) \quad \text{and} \quad i^! MHC_y((\mathcal{M}, F)) \,.
$$

In fact for the calculation of $i^! MHC_y((\mathcal{M}, F))$ we can even use the total complex of

(28)
$$gr_p^F V_{<-1} DR_{/A}^*(\mathcal{M})$$
$$gr_p^F(\nabla_t) \Big\downarrow$$
$$\Omega_A^1 \otimes gr_{p+1}^F V_{<0} DR_{/A}^*(\mathcal{M}) \,,$$

because the horizontal complexes of the quotient double complex

$$gr_p^F gr_{-1}^V DR_{/A}^*(\mathcal{M})$$
$$gr_p^F(\nabla_t) \Big\downarrow$$
$$\Omega_A^1 \otimes gr_{p+1}^F gr_0^V DR_{/A}^*(\mathcal{M})$$

are given by

$$gr_p^F gr_{-1}^V DR_{/A}^*(\mathcal{M}) \simeq gr_p^F DR_{/A}^*(gr_{-1}^V \mathcal{M})$$
$$\simeq i_*(\, gr_p^F DR_{/A}^*(gr_{-1}^V \mathcal{M}|M)\,)$$

and

$$gr_{p+1}^F gr_0^V DR_{/A}^*(\mathcal{M}) \simeq gr_{p+1}^F DR_{/A}^*(gr_0^V \mathcal{M})$$
$$\simeq i_*(\, gr_{p+1}^F DR_{/A}^*(gr_0^V \mathcal{M}|M)\,)\,,$$

if we make the identification $\Omega_A^1 \simeq \mathcal{O}_{M'} dt$. Here i is the closed inclusion $M \to M'$, with

$$F_p DR^*(gr_{-1}^V \mathcal{M}|M) = F_p DR^*(\Psi_{t,-1}(\mathcal{M}))$$

and

$$F_{p+1} DR^*(gr_0^V \mathcal{M}|M) = F_p DR^*(\Phi_{t,uni}(\mathcal{M}))$$

the absolute filtered de Rham complexes of the coherent left $\mathcal{D}_M$-modules $\Psi_{t,-1}(\mathcal{M})$ and $\Phi_{t,uni}(\mathcal{M})$ with its induced (shifted) filtrations F (which are good by assumption (s1.)). And from

$$gr_p^F DR^*(gr_\alpha^V \mathcal{M}|M) \in D_{coh,X}^b(M)$$

we get

$$gr_p^F gr_\alpha^V DR_{/A}^*(\mathcal{M}) \simeq i_*(\, gr_p^F DR^*(gr_\alpha^V \mathcal{M}|M)\,) \in D_{coh,X'}^b(M')\,,$$

together with

$$i^![gr_p^F gr_\alpha^V DR_{/A}^*(\mathcal{M})] = i^! i_*[gr_p^F DR^*(gr_\alpha^V \mathcal{M}|M)] = 0 \in G_0(X)$$

for $\alpha = -1$ and 0 by (22).

So also the total complex of

$$gr_p^F V_{<-1} DR_{/A}^*(\mathcal{M})$$

(29) $$gr_p^F(\nabla_t) \Big\downarrow$$

$$\Omega_A^1 \otimes gr_{p+1}^F V_{<0} DR_{/A}^*(\mathcal{M}) \, ,$$

has bounded coherent cohomology supported on X', and can be used for the calculation of $i^! MHC_y((\mathcal{M}, F))$. The advantage of this double complex is the fact, that it is only related to the nearby cycles $\Psi_{t,\alpha}(\mathcal{M})$ ($-1 \leq \alpha < 0$) and not to the unipotent vanishing cycles $\Phi_{t,uni}(\mathcal{M})$!

Next we want to calculate the class of its pullback $Li^*(-)$ in $G_0(X)$. But for the horizontal subcomplexes we get by Lemma 0.2:

(30)
$$Li^* gr_p^F V_{<-1} DR_{/A}^*(\mathcal{M}) \simeq i^* gr_p^F V_{<0} DR_{/A}^*(\mathcal{M})$$

$$\simeq gr_p^F DR(\, V_{<0}/V_{<-1}\mathcal{M}|M \,)$$

and

(31)
$$Li^* gr_{p+1}^F V_{<0} DR_{/A}^*(\mathcal{M}) \simeq i^* gr_{p+1}^F V_{<0} DR_{/A}^*(\mathcal{M})$$

$$\simeq gr_{p+1}^F DR(\, V_{<0}/V_{<-1}\mathcal{M}|M \,) \, .$$

Putting everything together, we get in $G_0(X)$ by additivity the equality:

(32)
$$i^! [gr_p^F DR^*(\mathcal{M})] =$$
$$-[gr_p^F DR^*(\, V_{<0}/V_{<-1}\mathcal{M}|M \,)] + [gr_{p+1}^F DR^*(\, V_{<0}/V_{<-1}\mathcal{M}|M \,)] \, .$$

Note that the $-$-sign for the first class on the right side is coming from the fact, that $gr_p^F DR(\, V_{<0}/V_{<-1}\mathcal{M}|M \,)$ viewed as a subcomplex of the double complex above agrees only up to a shift by one with the usual convention that "top-dimesional form" are in degree zero!

By using the filtration $V_\beta/V_{<-1}\mathcal{M}$ of $V_{<0}/V_{<-1}\mathcal{M}$ by $\mathcal{D}_{M'/A}$-modules ($-1 \leq \beta < 0$), we get by the assumption (s1.) and additivity the following equality in the Grothendieck group $G_0(X)$:

(33)
$$i^! [gr_p^F DR^*(\mathcal{M})] =$$
$$\sum_{-1 \leq \beta < 0} (\, -[gr_p^F DR^*(\, gr_\beta^V \mathcal{M}|M \,)] + [gr_{p+1}^F DR^*(\, gr_\beta^V \mathcal{M}|M \,)] \,) \, .$$

And this implies Theorem 0.1:

$$i^! MHC_y(\, (\mathcal{M}, F) \,) =$$

$$i^! (\, \sum_p [gr_{-p}^F DR(\mathcal{M})] \cdot (-y)^p \,) \ =$$

$$\sum_{p, -1 \leq \beta < 0} (\, -[gr_{-p}^F DR(\Psi_{t,\beta}\mathcal{M})] \cdot (-y)^p + [gr_{-(p-1)}^F DR(\Psi_{t,\beta}\mathcal{M})] \cdot (-y)^p \,)$$

$$= -(1 + y) \cdot \sum_{-1 \leq \beta < 0} MHC_y(\Psi_{t,\beta}(\mathcal{M}, F))$$
$$= -(1 + y) \cdot MHC_y(\Psi_t(\mathcal{M}, F)) .$$

$\square$

3. Specialization for Mixed Hodge modules

We can apply our first main theorem 2.1 in the context of algebraic mixed Hodge modules in the following more general context (for a base field $k \subset \mathbb{C}$). Let $f : X' \to \mathbb{A}^1_k$ be an algebraic function on the algebraic variety X' over k such that the inclusion of the zero fiber $i : X := \{f = 0\} \to X'$ is everywhere of codimension one (i.e. f is not vanishing on any irreducible component of X'). Then one gets as before a Gysin map $i^! : G_0(X') \to G_0(X)$ together with an exact nearby cycle transformation $\Psi_f : MHM(X'/k) \to MHM(X/k)$ on the abelian category of algebraic mixed Hodge modules. This induces therefore also a transformation of the corresponding Grothendieck groups

$$\Psi_f^H : K_0(MHM(X'/k)) \to K_0(MHM(X/k)) .$$

In the introduction we already explained that by [4] we have in this context a motivic Hodge-Chern class transformation

$$MHC_y : K_0(MHM(-)) \to G_0(-)[y, y^{-1}] ,$$

which *commutes with proper push down*. Let us recall our main result in this context.

Theorem 3.1. *This motivic Hodge-Chern class transformation commutes with specialization in the following sense:*

(34) $$- (1 + y) \cdot MHC_y(\Psi_f^H(-)) = i^! MHC_y(-)$$

as transformations $K_0(MHM(X'/k)) \to G_0(X)[y, y^{-1}]$.

Its proof can be reduced to theorem 2.1, since all transformations MHC_y, Ψ_f^H and $i^!$ commute with proper pushdown. By *resolution of singularities* and "additivity", $G_0(X')$ is generated by classes $\pi_*[\mathcal{M}]$ of mixed Hodge modules $\mathcal{M}$ on an algebraic manifold M, with $\pi : M \to X'$ a proper morphism. It is enough to prove the stated result for such a generating class $\pi_*[\mathcal{M}]$. If a connected component S of M maps into X, then this doesn't contribute to the specialization:

$$\Psi_f^H(\pi_*[\mathcal{M}|S]) = 0 \quad \text{and} \quad i^! MHC_y(\pi_*[\mathcal{M}|S]) = 0 .$$

So we can assume $f' := f \circ \pi$ is not vanishing on any connected component of M. Since all involved transformations MHC_y, Ψ_f^H and $i^!$ commute with π_*, we can assume $X' = M$ is smooth (with $\pi_* = id_*$). Using "additivity" and the weight filtration, one can even reduce to the case of pure Hodge modules. By using the (proper) graph embedding of f', we can reduce to the case $M' := M \times \mathbb{A}^1_k$ and t the projection onto $\mathbb{A}^1_k$ as studied before (with

$X' := graph_f(M) \subset M'$). Moreover the nearby cycles Ψ_t^H of the filtered $\mathcal{D}$-module $(\mathcal{M}, F)$ underlying a pure Hodge module on M' is exactly given as in definition 2.6. And $(\mathcal{M}, F)$ is by Saito [18] strictly specializable (as explained in remark 2.2). So in this case the claim is a special case of theorem 2.1. □

In the complex analytic context one doesn't have the full calculus of Grothendieck functors on the derived category of mixed Hodge modules. But nevertheless in the embedded context of an analytic subset $X \subset M$ in a complex manifold M, with $f : M \to \mathbb{C}$ a holomorphic function such that $X_0 := X \cap \{f = 0\}$ is *compact*, one has an *exact* nearby cycle functor

$$\Psi_f^H : MHM(X) \simeq MHM_X(M) \to MHM_{X_0}(M) \simeq MHM(X_0) \,,$$

with $MHM_X(M)$ the abelian category of mixed Hodge modules on M with support in X, and similarly for $MHM_{X_0}(M)$ (compare [19]). Also the graph embedding

$$g := (id_M, f) : M \to M' := M \times \mathbb{C} \quad \text{with} \quad g : X \simeq X' := g(X)$$

induces an exact functor $g_* : MHM_X(M) \to MHM_{X'}(M')$ commuting with Ψ_f^H and Ψ_t^H. Similarly one gets a motivic Hodge Chern class transformation

$$MHC_y : K_0(MHM(X)) = K_0(MHM_X(M)) \to G_0^X(M)[y, y^{-1}] \,.$$

Here one uses the fact that morphisms of analytic mixed Hodge modules are *strict* with respect to the (Hodge) filtration F of the underlying filtered $\mathcal{D}$-modules on M. And also MHC_y commutes with $g_* : G_0^X(M)[y, y^{-1}] \to G_0^{X'}(M')[y, y^{-1}]$.

So in this embedded complex analytic context we can argue as in the end of the argument before, and get the

Theorem 3.2. *Let $f : M \to \mathbb{C}$ be a holomorphic function on the complex manifold M, with $X \subset M$ a closed analytic subset. Assume $\{f = 0\}$ is a hypersurface of codimension one, with $X_0 := X \cap \{f = 0\}$ compact. Then the motivic Hodge-Chern class transformation MHC_y commutes with specialization in the following sense:*

$$(35) \qquad\qquad - (1 + y) \cdot MHC_y(\, \Psi_f^H(-) \,) = i^! MHC_y(-)$$

as transformations

$$K_0(MHM(X)) = K_0(MHM_X(M)) \to G_0^{X_0}(M)[y, y^{-1}] \,.$$

□

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